Sumudu Decomposition Method for Nonlinear Fractional Volterra Integro-Differential Equations

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The fractional integro-differential equations (FIDEs) are in general form of integer order integrodifferential equations. In this study concerns with the approximate analytical solution of the nonlinear Caputo fractional integro-differential equation of the following form:

$$D^{\alpha} y(t) = p(t)y(t) + g(t) + \int_0^t K(t,\tau)F(y(\tau))d\tau, \tag{1}$$

with the initial condition

$$y^{(i)}(0) = \beta_i; \quad i = 0,1,2,......,m-1, \tag{2}$$

where D^{α} is the Caputo fractional differential operator of order $\alpha, m-1 < \alpha \le m$, $f(t) \in L^2([0,1]), p(t) \in L^2([0,1])$ and $K(t,\tau) \in L^2([0,1]^2)$ are known functions, $y(t)$ is unknown functions.

Such kind of equations are the focus of research due to their pivotal role in the mathematical modeling of many physical problems in several fields of physics, engineering, and economics, such as arising in heat conduction in materials with memory, signal processing and fluid mechanics [1, 2, 3]. However, the fractional integro-differential equations are usually difficult to solve analytically and may not have exact or analytical solutions, so approximate and numerical methods for approximate solutions to integrodifferential equation of integer order are extended to solve fractional integro-differential equations.

In recent years, many methods have been developed to solve fractional integro-differential equations, especially nonlinear, which are receiving a lot of attention. This article aims to introduce a method for solving fractional nonlinear integro-differential equations, called the Sumudu decomposition method (SDM).

Fractional calculus

Definition 1 [4]. The Riemann-Liouville fractional integral of order $\alpha > 0$, of a real valued function $f(t)$ is defined as:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Definition 2 [4]. The fractional integral derivative of $f(t)$ in the Caputo's sense is defined as:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau.$$

for $m-1 < \alpha \le m,\ m \in \mathbb{N}$.

Sumudu transform

Definition 3 [5] The Sumudu transform over the following set of functions

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, \left| f(t) \right| < M e^{|t|/\tau_j} \ if\ t \in (-1)^j \times [0, \infty) \right\}$$

is defined for $u \in (-\tau_1, \tau_2)$ as

$$F(u) = S\left[f(t) \right] = \int_0^{\infty} e^{-t} f(ut) dt = \int_0^{\infty} \frac{1}{u} e^{-t/u} f(t) dt. \quad (3)$$

Function $f(t)$ in (3) is called inverse Sumudu transform of $F(u)$ and is denoted by $f(t) = S^{-1}\left[F(u) \right]$.

Some basic transform of the functions related to present work are as follow:

$$S\left[t^{\alpha} \right] = u^{\alpha} \Gamma(\alpha+1);\ \alpha > 0,$$

$$S^{-1}\left[u^{\alpha} \right] = \frac{t^{\alpha}}{\Gamma(\alpha+1)};\ \alpha > 0,$$

$$S\left[\int_0^t f(\tau)d\tau\right] = uS\left[f(t)\right].$$

Definition 4. Let $f(t)$ and $g(t)$ are continuous functions and exponential order, the convolution of $f(t)$ and $g(t)$ is defined as

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Theorem 1 [6] Let $f(t)$ and $g(t)$ are continuous function and exponential order. If $S\left[f(t)\right] = F(u)$ and $S\left[g(t)\right] = G(u)$ then

$$S\left[(f*g)(t)\right] = S\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = uF(u)G(u).$$

Theorem 2 [6] The Sumudu transform of the Caputo fractional derivative is defined as

$$S\left[D^\alpha f(t)\right] = u^{-\alpha}S\left[f(t)\right] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0),$$

for $m-1<\alpha<m,\ m\in\square$.

Analysis of the method

Firstly, we consider the fractional integro-differential equation of Volterra type. According to modified Sumudu decomposition method, we apply Sumudu transform first on both side of (1), we get

$$S\left[D^\alpha y(t)\right] = S\left[p(t)y(t)\right] + S\left[g(t)\right] + S\left[\int_0^t K(t,\tau)F(y(\tau))d\tau\right]. \qquad (4)$$

Using the property of Sumudu transform and simplifying, we can obtain

$$S\left[D^\alpha y(t)\right] = \sum_{k=0}^{m-1} u^k y^{(k)}(0) + u^\alpha S\left[p(t)y(t)\right] + u^\alpha S\left[g(t)\right] + u^\alpha S\left[\int_0^t K(t,\tau)F(y(\tau))d\tau\right]. \qquad (5)$$

Operating the inverse Sumudu transform on both sides of (5), we get

$$y(t)=S^{-1}\left[\sum_{k=0}^{m-1}u^k y^{(k)}(0)\right]+S^{-1}\left[u^\alpha S\left[p(t)y(t)\right]\right]+S^{-1}\left[u^\alpha S\left[g(t)\right]\right]+ \\ +S^{-1}\left[u^\alpha S\left[\int_0^t K(t,\tau)F(y(\tau))d\tau\right]\right]. \quad (6)$$

Next assume that

$$\begin{cases} f(t)=S^{-1}\left[\sum_{k=0}^{m-1}u^k y^{(k)}(0)\right]+S^{-1}\left[u^\alpha S\left[g(t)\right]\right], \\ R(y(t))=S^{-1}\left[u^\alpha S\left[p(t)y(t)\right]\right], \\ N(y(t))=S^{-1}\left[u^\alpha S\left[\int_0^t K(t,\tau)F(y(\tau))d\tau\right]\right]. \end{cases} \quad (7)$$

Thus, equation (6) can be written in the following form

$$y(t)=f(t)+R(y(t))+N(y(t)), \quad (8)$$

where f is a known function, R and N are given linear and nonlinear operator of y ,respectively.

The second step in modified Sumudu decomposition method is that we represent solution as in form of infinite series, given as follow:

$$y(t)=\sum_{n=0}^{\infty}y_n,$$

where the term y_n are to be recursively computed.

Thus, equation (8) is given as

$$\sum_{n=0}^{\infty}y_n=f(t)+\sum_{n=0}^{\infty}R(y_n)+\sum_{n=0}^{\infty}N(y_n),$$

then a recurrence relation is defined as follow:

$$y_0=f(t), \\ y_{n+1}=R(y_n)+N(y_n). \quad (9)$$

Applications

Example 1. Consider the following nonlinear fractional Volterra integro- differential equation:

$$D^{\alpha} y(t) = y^2(t) + y(t) - 1 - 2\int_0^t y^2(\tau) d\tau, \tag{10}$$

$$0 < \alpha \le 1,$$

with the following initial condition

$$y(0) = 1, \tag{11}$$

which has the exact solution in the case of $\alpha = 1$ is $y(t) = e^t$.

To solve this problem by the proposed method, we apply Sumudu transform on both side of (10) and using the inverse Sumudu transform, we have

$$f(t) = S^{-1}[1] + S^{-1}\left[u^{\alpha} S[-1]\right],$$

$$R(y(t)) = S^{-1}\left[u^{\alpha} S\left[y^2(t) + y(t)\right]\right],$$

$$N(y(t)) = -2S^{-1}\left[u^{\alpha+1} S\left[y^2(t)\right]\right].$$

Then, using (9) we get

$$y_0 = f(t),$$

$$y_1 = R(y_0) + N(y_0) = S^{-1}\left[u^{\alpha} S\left[y_0^2 + y_0\right]\right] - 2S^{-1}\left[u^{\alpha+1} S\left[y_0^2\right]\right],$$

$$y_2 = R(y_1) + N(y_1) = S^{-1}\left[u^{\alpha} S\left[y_1^2 + y_1\right]\right] - 2S^{-1}\left[u^{\alpha+1} S\left[y_1^2\right]\right],$$

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$$y_{n+1} = R(y_n) + N(y_n) = S^{-1}\left[u^{\alpha} S\left[y_n^2 + y_n\right]\right] - 2S^{-1}\left[u^{\alpha+1} S\left[y_n^2\right]\right],$$

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$$y(t) = y_0 + y_1 + y_2 + \dots\dots\dots$$

Hence, the 2-term approximate solution of problem (10) and (11) is

$$y_0 = f(t) = 1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$y_1 = \frac{2t^{\alpha}}{\Gamma(\alpha+1)}\left[1-\frac{t}{\alpha+1}\right] + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left[\frac{4t}{2\alpha+1}-3\right] + \frac{t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}\left[1-\frac{2t}{3\alpha+1}\right],$$

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This results from the SDM when $\alpha = 1$ match the exact solution

$$y(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + ... = e^t.$$

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