Solving Nonlinear Fractional Differential Equations using a Decomposition Method

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Over the last years, fractional differential equations have increased much consideration because of broad utilization in the mathematical modelling of physical problems. These are a generalization of classical integer order ordinary differential equations, are increasingly used to address the needs of problems in fluid mechanics, biology, engineering and other applications. It is not obvious that an exact solution of these problem types could be calculated. Generally, the numerical solutions can be derived. Various methods have been employed to solve fractional differential equations. As example, Laplace transform method, Fourier transforms method , Adomain decomposition method, and the new transform method [1-4]. The aim of the present paper is to use the Decomposition Method (DM), in order to provide explicit approximate solutions for further nonlinear fractional initial value problems.

Building on its valuable properties, the proposed transform has as of now demonstrated much efficacy. It is uncovered that it can take care of nonlinear differential problems resulting from some physical issues.

Sumudu transformation

$$f(x) \Rightarrow F(u)$$

$$F(u) = S\left[f(x)\right] = \int_0^\infty f(ux)e^{-x}dx. \tag{1}$$

$$S\left[x^\beta\right] = u^\beta \Gamma(\beta+1), \tag{2}$$

$$S\left[f^{(n)}(x)\right] = \frac{F(u)}{u^n} - \frac{f(0)}{u^n} - \frac{f^{'}(0)}{u^{n-1}} - .... - \frac{f^{(n-1)}(0)}{u}, \tag{3}$$

$$S^{-1}\left[u^{\beta-1}\right]=\frac{x^{\beta-1}}{\Gamma(\beta)}. \tag{4}$$

Algorithm of the Method

$$D^{\alpha}y(x)+a_n y^{(n)}(x)+a_{n-1}y^{(n-1)}(x)+.......+a_0 y(x)+F(y(x))=f(x) \tag{5}$$

Subject to the initial conditions,

$$y^{(k)}=b_k,\quad k=\overline{0,n-1},\qquad n-1<\alpha\le n.$$

Where a_k, b_k are known real constants, F is a nonlinear operator and $f(x)$ is known function. Taking the transform (1) of the equation (5), to obtain,

$$S[D^{\alpha}y(x)]+a_n S[y^{(n)}(x)]+a_{n-1}S[y^{(n-1)}(x)]+.......+a_0 S[y(x)]+S[F(y(x))]=S[f(x)]$$

Applying the formula of the transform (1) and using (2),(3) we fix;

$$S[y(x)]=\left[u^{\alpha}S[f(x)]+\sum_{k=0}^{n-1}\frac{b_k}{u^{n-k-\alpha}}\right]- \\ -u^{\alpha}\left[a_n S[y^{(n)}(x)]+a_{n-1}S[y^{(n-1)}(x)]+.......+a_0 S[y(x)]+S[F(y(x))]\right] \tag{6}$$

The new decomposition method represents the solution as an infinite series

$$y(x)=\sum_{i=0}^{\infty}y_i(x) \tag{7}$$

and the nonlinear term $F(y(x))$ decomposes as;

$$F(y(x))=\sum_{i=0}^{\infty}A_i(x) \tag{8}$$

where,

$$A_i(x)=\frac{1}{i!}\frac{d^i}{dp^i}F\left(\sum_{i=0}^{\infty}p^i y_i(x)\right) \tag{9}$$

are Adomian polynomials. Substituting (7), (8) and (9), into (6), to get;

$$S\left[\sum_{i=0}^{\infty}y_i(x)\right]=\left[u^{\alpha}S[f(x)]+\sum_{k=0}^{n-1}\frac{b_k}{u^{n-k-\alpha}}\right]- \\ -u^{\alpha}\left[a_n S\left[\sum_{i=0}^{\infty}y_i^{(n)}(x)\right]+a_{n-1}S\left[\sum_{i=0}^{\infty}y_i^{(n-1)}(x)\right]+...+a_0 S\left[\sum_{i=0}^{\infty}y_i(x)\right]+S\left[\sum_{i=0}^{\infty}A_i(x)\right]\right]$$

The iterations are defined by the recursive relations;

$$S[y_0(x)] = \left[u^\alpha S[f(x)] + \sum_{k=0}^{n-1} \frac{b_k}{u^{n-k-\alpha}}\right] \tag{10}$$

$$S[y_i(x)] = -u^\alpha \left[a_n S[y_{i-1}^{(n)}(x)] + a_{n-1} S[y_{i-1}^{(n-1)}(x)] + ... + a_0 S[y_{i-1}(x)] + S[A_{i-1}(x)]\right] \tag{11}$$

Numerical Results

Example: Look at the following fractional nonlinear equation,

$$D^\alpha y(x) = y^2(x) - 2y(x) + 1 \tag{12}$$

with the initial condition,

$$y(0) = 0 \tag{13}$$

Solution: Take the transform (1) and using (2),(3) of the equation (12) and use the initial conditions (13), and so we suffer:

$$\frac{1}{u^\alpha} S[y(x)] = S\left[y^2(x)\right] - 2S[y(x)] + 1$$

$$\frac{1}{u^\alpha} S[y(x)] = S\left[F(y(x))\right] - 2S[y(x)] + 1$$

$$F(y(x)) = \sum_{i=0}^{\infty} A_i(x)$$

The new decomposition series (10),(11) has the form,

$$S\left[y_0(x)\right] = u^\alpha,$$

$$S\left[y_i(x)\right] = u^\alpha S\left[\sum_{i=0}^{\infty} A_{i-1}(x)\right] - 2u^\alpha S\left[\sum_{i=0}^{\infty} y_{i-1}(x)\right]$$

where:

$$A_i(x) = \frac{1}{i!}\frac{d^i}{dp^i}\left[y_0^2(x) + 2py_0(x)y_1(x) + p^2\left(y_1^2(x) + 2y_0(x)y_2(x)\right) + ...\right]_{p=0}$$

Then we can find;

$$y_0(x) = S^{-1}\left[u^\alpha\right] = \frac{x^\alpha}{\Gamma(\alpha+1)},$$

$$S\left[y_1(x)\right] = u^\alpha S\left[A_0(x)\right] - 2u^\alpha S\left[y_0(x)\right], \text{ where } A_0(x) = y_0^2(x)$$

$$S\left[y_1(x)\right] = u^\alpha S\left[\frac{x^{2\alpha}}{\Gamma^2(\alpha+1)}\right] - 2u^\alpha S\left[\frac{x^\alpha}{\Gamma(\alpha+1)}\right] = \frac{u^{3\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - 2u^{2\alpha}$$

$$y_1(x)=\frac{x^{3\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}-\frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}$$

$S[y_2(x)]=u^\alpha S[A_1(x)]-2u^\alpha S[y_1(x)]$, where $A_1(x)=2y_0(x)y_1(x)$

$$S[y_2(x)]=u^\alpha S\left[\frac{2x^\alpha}{\Gamma(\alpha+1)}\left(\frac{x^{3\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}-\frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}\right)\right]-2u^\alpha S\left[\frac{x^{3\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}-\frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}\right]$$

$$y_2(x)=\frac{2x^{5\alpha}\Gamma(4\alpha+1)\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}-\frac{4x^{4\alpha}\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}-\frac{2x^{4\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)}-\frac{4x^{3\alpha}}{\Gamma(3\alpha+1)}$$

Then we have;

$$y(x)=\frac{x^\alpha}{\Gamma(\alpha+1)}+\frac{x^{3\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}-\frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}+$$
$$+\frac{2x^{5\alpha}\Gamma(4\alpha+1)\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}-\frac{4x^{4\alpha}\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}-\frac{2x^{4\alpha}\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)}-\frac{4x^{3\alpha}}{\Gamma(3\alpha+1)}+.........$$

If $\alpha=1$ we obtain:

$$y_0(x)=x,\quad y_1(x)=\frac{x^3}{3}-x^2 \text{ and } y_2(x)=\frac{2x^5}{15}-\frac{2x^4}{3}+\frac{2x^3}{3}$$

This results from the DM when $\alpha=1$ match the exact solution

$$y(x)=x-x^2+x^3-.........=1-\frac{1}{1+x}$$

Thus, the proposed method is a very effective and accurate method that can be utilized to provide analytical results for nonlinear fractional differential equations.

References

[1] Самко С.Г., Килбас А.А., Маричев О.И. Интегралы и производные дробного порядка и некоторые их приложения. Мн: Наука и техника.—1987.— 688 с.

[2] Miller K. S., Ross B., 1993, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons. Inc., New York

[3] T. M. Elzaki, S. M. Elzaki, E. M. A. Hilal, Elzaki and Sumudu transforms for solving some differential equations. Global JPure Appl Mathe 2012;8(2):167-73.

[4] Saeed Kazem, Exact solution of some linear fractional differential equation by Laplace transform,International Journal of Nonlinear Science Vol.16(2013) No.1,pp. 3-11.