SHTOLS TEOREMASI VA UNING TADBIQLARI

Muzaffarova Mohinur Umarovna

Buxoro davlat universiteti

Fizika-matematika fakulteti talabasi

Annotatsiya. Maqolada Shtols teoremasining isboti batafsil yoritilgan. Bir qator murakkab ketma-ketliklarning limitini hisoblashda undan keng foydalanilgan.

Kalit so'zlar. Haqiqiy sonlar ketma- ketligi, ayniyat, cheklangan limit, limit, cheksizlik.

STOLS THEOREM AND ITS APPLICATIONS

Muzaffarova Mohinur Umarovna

Bukhara State University

Student of the Faculty of Physics and Mathematics

Annotation. The article describes the proof of Stolz's theorem in detail. It is widely used in calculating the limit of a number of complex sequences.

Keywords. Sequence of real numbers, arithmetic, finite limit, limit, infinity.

Shtols teoremasi haqiqiy sonlar ketma-ketligi limitini topishga yordam beradi. Teorema 1885-yilda avstriyalik matematik Otto Shtols tomonidan nashr etilgan va uning nomi bilan atalgan. Tabiatan Shtols teoremasi Lopital qoidasining diskret analogidir.

Aytish joizki, Shtols teoremasining isboti klassik adabiyotlarda keltirilmagan. Ayrim masalalar to'plamlarida misol sifatida taqdim qilingan. Masalan, B.P.Demidovichning «Sbornik zadach i uprajneniy po matematicheskomu analize» [1] nomli kitobining 25 betida 143-misol sifatida berilgan. Shuning uchun maqolada uning isboti batafsil keltirildi va teorema yordamida bir qator murakkab ketma-ketliklarning limiti hisoblab ko'rsatildi.

Teorema (Shtols). a_n va b_n ikkita haqiqiy sonlar ketma — ketligi bo'lsin va b_n musbat, chegaralanmagan va qat'iy ravishda (hech bo'lmaganda ma'lum bir sondan boshlab) o'sib boradi. Agar

$$\lim_{n\to\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}}$$

mavjud bo'lsa, u holda quyidagi limit ham mavjud

$$\lim_{n\to\infty}\frac{a_n}{b_n}$$

bo'ladi va

$$\lim_{n\to\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}}=\lim_{n\to\infty}\frac{a_n}{b_n}.$$

Teoremaninig isboti [2] va [3] adabiyotlarda kitobida keltirilgan.

Isbot. Aytaylik, avval limit cheklangan L songa teng. Demak, ixtiyoriy berilgan har qanday $\varepsilon>0$ uchun shunday $n>0$ mavjud bo'ladiki, $n>N$ lar uchun

$$L-\frac{\varepsilon}{2}<\frac{a_n-a_{n-1}}{b_n-b_{n-1}}<L+\frac{\varepsilon}{2}$$

uchun quyidagi o'rinli bo'ladi

Shunday qilib, har qanday $n>N$ lar uchun barcha nisbatlar

$$\frac{a_{N+1}-a_N}{b_{N+1}-b_N},\frac{a_{N+2}-a_{N+1}}{b_{N+2}-b_{N+1}},\cdots,\frac{a_n-a_{n-1}}{b_n-b_{n-1}}$$

shu sonlar orasida yotadi.

Ushbu nisbatlarning maxrajlari musbat bo'lganligi sababli (b_n ketma-ketligining qat'iy o'sishi tufayli), mediana xossasiga ko'ra, xuddi shu limitlar orasida ham nisbat mavjud:

$$\frac{a_n-a_N}{b_n-b_N}$$

uning surati yuqorida yozilgan kasrlar suratining yig'indisi, maxraji esa barcha maxrajlarning yig'indisidir. Shunday qilib, $n>N$ bo' lganda:

$$\left|\frac{a_n - a_N}{b_n - b_N} - L\right| < \frac{\varepsilon}{2}$$

o'rinli bo'ladi.

Endi quyidagi ayniyatni ko'rib chiqamiz (to'g'ridan-to'g'ri tekshiriladi):

$$\frac{a_n}{b_n} - L = \frac{a_N - L\, b_N}{b_n} + \left(1 - \frac{b_N}{b_n}\right)\left(\frac{a_n - a_N}{b_n - b_N} - L\right),$$

bundan

$$\left|\frac{a_n}{b_n} - L\right| = \left|\frac{a_N - L\, b_N}{b_n}\right| + \left|\frac{a_n - a_N}{b_n - b_N} - L\right|$$

ekanligi kelib chiqadi.

$n > N$ bo'lganda, birinchi qo'shiluvchi $\frac{\varepsilon}{2}$ dan kichik bo'ladi. $n > M$ bo'lganda, ikkinchi qo'shiluvchi ham $\frac{\varepsilon}{2}$ dan kichik bo'ladi. Bu yerda $M -$ yetarlicha katta son va $b_n \to +\infty$ o'rinli. Agar $M > N$ deb olsak, unda $n > M$ bo'lganda

$$\left|\frac{a_n}{b_n} - L\right| < \varepsilon$$

ega bo'lamiz. Teorema isbotlandi.

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = +\infty,$$

bo'lsin. Bundan kelib chiqadiki, yetarlicha katta n bo'lganda:

$$a_n - a_{n-1} > b_n - b_{n-1} \;\text{ va }\; \lim_{n\to\infty} a_n = +\infty$$

a_n ketma-ketlik qat'iy ravishda oshadi (ma'lum bir sondan boshlab). Bu holda, teoremaning isbotlangan qismiga $\frac{b_n}{a_n}$ teskari munosabatiga qo'llanilishi mumkin:

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{b_n - b_{n-1}}{a_n - a_{n-1}} = 0,$$

bu yerdan kelib chiqadi:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = +\infty$$

Agar limit $-\infty$ bo'lsa, u holda $\{-a_n\}$ ketma-ketlik ko'rib chiqiladi.

1-misol. Shtols teoremasidan foydalanib quyidagi limitni hisoblang:

$$\lim_{n\to\infty}\frac{(n+1)(n+2)(n+3)}{n^3}=\left[\frac{\infty}{\infty}\right]=$$

$$=\lim_{n\to\infty}\frac{(n^2+2n+n+2)(n+3)-n(n+1)(n+2)}{n^3-(n-1)^3}=$$

$$=\lim_{n\to\infty}\frac{n^3+3n^2+3n^2+9n+2n+6-(n^2+n)(n+2)}{n^3-n^3+3n^2-3n+1}=$$

$$=\lim_{n\to\infty}\frac{n^3+6n^2+11n+6-n^3-2n^2-n^2-2n}{3n^2-3n+1}=\lim_{n\to\infty}\frac{3n^2+9n+6}{3n^2-3n+1}=$$

$$=\lim_{n\to\infty}\frac{3n^2+9n+6-(3(n-1)^2+9(n-1)+6)}{3n^2-3n+1-(3(n-1)^2-3(n-1)+1)}=$$

$$=\lim_{n\to\infty}\frac{3n^2+9n+6-(3(n-1)^2+9(n-1)+6)}{3n^2-3n+1-(3(n-1)^2-3(n-1)+1)}$$

$$=\lim_{n\to\infty}\frac{3n^2+9n+6-(3n^2-6n+3+9n-9+6)}{3n^2-3n+1-(3n^2-6n+3-3n+3+1)}=$$

$$=\lim_{n\to\infty}\frac{3n^2+9n+6-3n^2+6n-3-9n+9-6)}{3n^2-3n+1-3n^2+6n-3+3n-3-1}=$$

$$=\lim_{n\to\infty}\frac{6n+6}{6n-6}=\lim_{n\to\infty}\frac{6n+6-(6(n-1)+6)}{6n-6-(6(n-1)-6)}=$$

$$=\lim_{n\to\infty}\frac{6n+6-6n+6-6}{6n-6-6n+6+6}=\frac{6}{6}=1$$

2-misol. Quyidagi funksiyaning limitsini toping:

$$\lim_{n\to\infty}\frac{2n+1}{n+1}$$

1-usul. Funksiyani surat va maxrajidan n ni chiqarib olamiz

$$\lim_{n\to\infty}\frac{2n+1}{n+1}=\left[\frac{\infty}{\infty}\right]=\lim_{n\to\infty}\frac{n(2+\frac{1}{n})}{n(1+\frac{1}{n})}=\lim_{n\to\infty}\frac{2+\frac{1}{n}}{1+\frac{1}{n}}=\frac{2}{1}=2.$$

Bu yerda

$$\lim_{n\to\infty}\frac{1}{n}=0.$$

2-usul. Biz bu usulda Shotls teoremasidan foydalanamiz:

$$\lim_{n\to\infty}\frac{2n+1}{n+1}=\left[\frac{\infty}{\infty}\right]=\lim_{n\to\infty}\frac{2n+1-(2(n-1)+1)}{n+1-((n-1)+1)}=$$

$$=\lim_{n\to\infty}\frac{2n+1-(2n-2+1)}{n+1-(n-1+1)}=\lim_{n\to\infty}\frac{2n+1-2n+2-1}{n+1-n+1-1}=\frac{2}{1}=2$$

3-misol. Quyidagi ketma-ketlikning limitini toping:

$$\lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3}=\lim_{n\to\infty}\frac{n^2}{n^3-(n-1)^3}=$$

$$\lim_{n\to\infty}\frac{n^2}{n^3-n^3+3n^2-3n+1}=\lim_{n\to\infty}\frac{n^2}{3n^2-3n+1}=$$

$$=\lim_{n\to\infty}\frac{n^2-(n-1)^2}{3n^2-3n+1-(3(n-1)^2-3(n-1)+1)}=$$

$$=\lim_{n\to\infty}\frac{n^2-n^2+2n-1}{3n^2-3n+1-(3n^2-6n+3-3n+3+1)}=$$

$$=\lim_{n\to\infty}\frac{2n-1}{3n^2-3n+1-3n^2+6n-3+3n-3-1}=$$

$$=\lim_{n\to\infty}\frac{2n-1}{6n-6}=\lim_{n\to\infty}\frac{2n-1-(2(n-1)-1)}{6n-6-(6(n-1)-6)}=$$

$$=\lim_{n\to\infty}\frac{2n-1-2n+2+1}{6n-6-6n+6+6}=\frac{2}{6}=\frac{1}{3}.$$

FOYDALANILGAN ADABIYOTLAR RO'YHATI

1. Demidovich B.P. «Sbornik zadach i uprajneniy po matematicheskomu analize». Moskva, CheRo, 1997 g., 624 s.

2. Fixtengol's G.M. Kurs differensial'nogo i integral'nogo ischisleniya. — M.: Fizmatlit, 2003 g., t. 1. str. 78-79.

3. Arxipov G.I., Sadovnichiy V.A., Chubarikov V.N. Leksii po matematicheskomu analizu. Moskva, Visshaya shkola, 1999 g., str. 43-44