ON THE BEHAVIOR OF SOLUTIONS FOR A SYSTEM OF MULTIDIMENSIONAL HEAT TRANSFER EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Consider the following nonlinear system of parabolic equations coupled via nonlinear boundary conditions

$$\begin{cases} u_t = \nabla\left(\left|\nabla u^{m_1}\right|^{p_1-2}\nabla u^{m_1}\right), & x\in R_+^N,\ t>0, \\ u_t = \nabla\left(\left|\nabla \upsilon^{m_2}\right|^{p_2-2}\nabla \upsilon^{m_2}\right), & x\in R_+^N,\ t>0, \end{cases} \tag{1}$$

$$\begin{cases} -\left|\nabla u^{m_1}\right|^{p_1-2}\dfrac{\partial u^{m_1}}{\partial x_1} = u^{\beta_1}(0,t)\upsilon^{q_1}(0,t), & x_1=0,\ t>0, \\ -\left|\nabla \upsilon^{m_2}\right|^{p_2-2}\dfrac{\partial \upsilon^{m_2}}{\partial x_1} = u^{\beta_2}(0,t)\upsilon^{q_2}(0,t), & x_1=0,\ t>0, \end{cases} \tag{2}$$

$$u(x,0)=u_0(x),\ \upsilon(x,0)=\upsilon_0(x),\ x>0, \tag{3}$$

where $R_+^N=\{(x_1,x')\,|\,x'\in R^{N-1},\ x_1>0\}$, $m_i>1$, $p_i>1+1/m_i$, $\beta_i>0$, $q_i>0\,(i=1,2)$, u_0 and $\upsilon_0(x)$ are nonnegative continuous functions with compact supports in R_+^N.

The nonlinear parabolic system of equations (1) occurs in various applications as a model of biological populations, chemical reactions, heat transfer, filtration, diffusion, etc. For example, $u(x,t)$ and $\upsilon(x,t)$ are the densities of two biological populations in the process of migration or the temperatures of two porous materials during heat propagation [1-5].

The nonlinear boundary conditions (2) can be used to describe the influx of energy input at the boundary. For instance, in the heat transfer process (2) represents the heat flux, and hence the boundary conditions represent a nonlinear radiation law at the boundary. This kind of boundary conditions appears also in combustion problems when the reaction happens only at the boundary of the container, for example because of the presence of a solid catalyzer, see [1, 4] for a justification.

Equations (1) under the conditions $p_i>1+1/m_i$ $(i=1,2)$ correspond to the case of slow diffusion, and under $p_i<1+1/m_i$ $(i=1,2)$ the case of fast diffusion. In the case of slow diffusion, equations (1) are degenerated; it is well known that degenerate equations need not possess classical solutions. However, the local in time existence of the weak solution to the problem (1)-(3), defined in the usual integral way [1, 4, 5].

In recent years have been intensively studied the problems on blow-up and global existence conditions, blow-up rates to nonlinear parabolic equations [6-10]. In particular, critical exponents of the Fujita type, which plays an important role in studying the properties of mathematical models of various nonlinear processes, are described by nonlinear parabolic equations and a system of such equations of mathematical physics (see [1-4] and references therein).

Quirós and Rossi [6] considered the problem (1)-(3) in the case $\beta_1 = 0$, $q_2 = 0$, $p_i = 2\ \ (i = 1,2)$, with notations

$$\lambda_1 = \frac{2q_1 + m_2 + 1}{(m_1+1)(m_2+1) - 4q_1q_2},\ \lambda_2 = \frac{2q_2 + m_1 + 1}{(m_1+1)(m_2+1) - 4q_1q_2},$$

$$\chi_1 = \frac{q_1(m_1 - 1 - 2q_2) + (m_2+1)m_1}{(m_1+1)(m_2+1) - 4q_1q_2},\ \chi_2 = \frac{q_2(m_2 - 1 - 2q_1) + (m_1+1)m_2}{(m_1+1)(m_2+1) - 4q_1q_2}.$$

They proved that the critical Fujita exponents to (1)-(3) are described by $\lambda_i + \chi_i = 0,\ (i = 1,2)$, while the blow-up rate of the positive solution is $O\left((T-t)^{\lambda_1}\right)$ for component u and $O\left((T-t)^{\lambda_2}\right)$ for υ as $t \to T$.

In [7], Yongsheng Mi, Chunlai Mu, and Botao Chen considered the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial u}{\partial x}\right|^{p_1-2}\dfrac{\partial u^{m_1}}{\partial x}\right), & x > 0,\ 0 < T < 0, \\ \dfrac{\partial \upsilon}{\partial t} = \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial \upsilon}{\partial x}\right|^{p_2-2}\dfrac{\partial \upsilon^{m_2}}{\partial x}\right), & x > 0,\ 0 < T < 0, \end{cases}$$

$$\begin{cases} -\left|\dfrac{\partial u}{\partial x}\right|^{p_1-2}\dfrac{\partial u^{m_1}}{\partial x}\Bigg|_{x=0} = \upsilon^{q_1}(0,t), & 0 < T < 0, \\ -\left|\dfrac{\partial \upsilon}{\partial x}\right|^{p_2-2}\dfrac{\partial \upsilon^{m_2}}{\partial x}\Bigg|_{x=0} = u^{q_2}(0,t), & 0 < T < 0, \end{cases}$$

$$u(x,0) = u_0(x),\ \upsilon(x,0) = \upsilon_0(x),\ \ x > 0.$$

They showed that the critical global existence exponent and critical Fujita exponent are $q_1q_2 = \dfrac{(2p_1 - 1 + m_1)(2p_2 - 1 + m_2)}{p_1p_2}$ and $\min\{l_1 - k_1, l_2 - k_2\} = 0$, where

$$k_1 = \frac{(p_2-1)p_1q_1 + (p_1-1)(2q_2 + m_2 + 1)}{q_1q_2p_1p_2 - (2q_1 + m_1 + 1)(2q_2 + m_2 + 1)},$$

$$k_2 = \frac{(p_1 - 1) p_2 q_2 + (p_2 - 1)(2q_1 + m_1 + 1)}{q_1 q_2 p_1 p_2 - (2q_1 + m_1 + 1)(2q_2 + m_2 + 1)},$$

$$l_1 = \frac{k_2 q_1 - k_1 (p_1 + m_1 - 2)}{p_1 - 1}, \; l_2 = \frac{k_1 q_2 - k_2 (p_2 + m_2 - 2)}{p_2 - 1}.$$

Zhaoyin Xiang, Chunlai Mu and Yulan Wang [8] studied the problem (1)-(3) in one dimensional case, when $\beta_1 = 0$, $q_2 = 0$. They proved that the solutions of (1)-(3) are global if $q_1 \beta_2 \le \frac{(p_1 - 1)(p_2 - 1)(m_1 + 1)(m_2 + 1)}{p_1 p_2}$, and may blow up in finite time if $q_1 \beta_2 > \frac{(p_1 - 1)(p_2 - 1)(m_1 + 1)(m_2 + 1)}{p_1 p_2}$. In the case of $q_1 \beta_2 > \frac{(p_1 - 1)(p_2 - 1)(m_1 + 1)(m_2 + 1)}{p_1 p_2}$, if $y_1 - r_1 \le 0$ or $y_2 - r_2 \le 0$, where

$$r_1 = \frac{q_1 p_1 (p_2 - 1) + (p_1 - 1)(p_2 - 1)(m_2 + 1)}{q_1 \beta_2 p_1 p_2 - (p_1 - 1)(p_2 - 1)(m_1 + 1)(m_2 + 1)},$$

$$r_2 = \frac{q_2 p_2 (p_1 - 1) + (p_1 - 1)(p_2 - 1)(m_1 + 1)}{q_1 \beta_2 p_1 p_2 - (p_1 - 1)(p_2 - 1)(m_1 + 1)(m_2 + 1)}, \quad y_1 = \frac{q_1 r_2 - m_1 r_1 (p_1 - 1)}{p_1 - 1},$$

$y_2 = \frac{q_2 r_1 - m_2 r_2 (p_2 - 1)}{p_2 - 1}$, then every non-negative, non-trivial solutions of (1)-(3) blow up in finite time.

The purpose of this study is to find the conditions of existence and nonexistence of solutions to problem (1)-(3) over time based on self-similar analysis. Various self-similar solutions to the problem (1)-(3) are constructed, estimates of the solutions are obtained, the critical Fujita exponents and critical exponents for the global existence of the solution are established.

We introduce the following notation

$$\alpha_1 = \frac{\beta_1 p_1 (p_2 - 1) + (p_1 - 1)}{q_1 \beta_2 p_1 p_2 - s_1 s_2}, \; \alpha_2 = \frac{\beta_2 p_2 (p_1 - 1) + p_2 - 1}{q_1 \beta_2 p_1 p_2 - s_1 s_2}$$

$$\lambda_1 = \frac{1 - \alpha_1 (m_1 (p_1 - 1) - 1)}{p_1}, \; \lambda_2 = \frac{1 - \alpha_2 (m_2 (p_2 - 1) - 1)}{p_2}$$

$$s_1 = (p_1 - 1)(m_1 + 1) - \beta_1 p_1, \; s_2 = (p_2 - 1)(m_2 + 1) - q_2 p_2.$$

Theorem 1. Assume that $\beta_1 \le \frac{(p_1-1)(m_1+1)}{p_1}$, $q_2 \le \frac{(p_2-1)(m_2+1)}{p_2}$. If $q_1\beta_2 \le \left(\frac{(p_1-1)(m_1+1)}{p_1}-\beta_1\right)\left(\frac{(p_2-1)(m_2+1)}{p_2}-q_2\right)$, then every solution of the system (1)-(3) exists globally in time.

Theorem 2. Assume that $\beta_1 \le \frac{(p_1-1)(m_1+1)}{p_1}$, $q_2 \le \frac{(p_2-1)(m_2+1)}{p_2}$ and $q_1\beta_2 > \left(\frac{(p_1-1)(m_1+1)}{p_1}-\beta_1\right)\left(\frac{(p_2-1)(m_2+1)}{p_2}-q_2\right)$. If $\min\{N\lambda_1-\alpha_1, N\lambda_2-\alpha_2\}>0$, then solutions of the system (1)-(3) with small initial data globally in time.

Theorem 5 is proved by the method described in [17].

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