Solving fractional differential equations using the Gamma function

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Fractional integration and differentiation is a rapidly developing area of modern analysis, which has a long history and rich content, due to penetration and relationships with various issues of the theory of functions, integral and differential equations, functional analysis, special functions and integral transformations. Generalization of the concept of differentiation $\frac{d^p f(t)}{dt^p}$ to non-integer values p, arose from the very beginning of differential calculus. The first steps were taken by J.I. Euler in 1738, P. Laplace in 1812, J. Fourier in 1822. The actual history of fractional calculus should be traced back to the works of N.H.Abel and J.Liouville, who appeared in the 30s of the 19th century. Next to the works of J.Liouville in importance should be placed the works of B. Riemann, who came to the construction of fractional integration, which has since served as one of the main forms of fractional integration.

The history of the development of fractional integrodifferentiation includes many works, in which already known results were rediscovered at different times, sometimes with the same means as the previous ones, and sometimes based on other methods. This circumstance was aggravated by the fact that there are a large number of different approaches to fractional integro-differentiation and different directions in fractional calculus. Comparisons between these approaches and trends have been rare and relatively little is known. An important step in development was the writing of a book that combined various studies in the direction of studying fractional derivatives and integrals, written by S.G. Samko, A.A. Kilbas and O.I. Marichev [1]. A few years later, a book appeared by Miller K., Ross V. [2]. The further development of fractional calculus was promoted by AM Nakhushev's book [3]. This monograph presents the thoughts and ideas that the author had in the process of searching for methods for solving various, both local and nonlocal initial, mixed and boundary value problems for partial differential equations of basic and qualitatively new types. A.V. Pskhu [4] conducts a study of linear equations with two independent variables of order less than or equal to unity. The cited books contain an extensive list of publications devoted to differential equations with fractional derivatives. Among them are the works of Aleroev T.S. [5], [6], Voroshilova A.A. and Kilbasa A.A. [7], Gekkieva S.Kh. [8], Arendt W. [9], Schneider WR, Wyss W. [10], etc.

Euler's gamma functions are important in the theory of fractional integration and differentiation since they are a generalization of the factorial concept for non-natural numbers. The beta function is generally defined in terms of the gamma function. The psi function is the logarithmic derivative of the gamma function.

Let $z \in C$. Gamma function $\Gamma(z)$ was defined by Euler as the limit [3]

$$\Gamma(z) = \lim_{N \to \infty} \frac{N!N^z}{z(z+1)(z+2)(z+N)}, \quad z \in C,$$

but, more often the definition is used in the form of an Euler integral of the second kind

$$\Gamma(z) = \int_0^\infty y^{z-1} e^{-y} dy, \quad \operatorname{Re} z > 0, \tag{1}$$

which converges for all $z \in C$, for which $\operatorname{Re} z > 0$.

Integration by parts of the expression (1) leads to the recurrent formula

$$\Gamma(z+1) = z\Gamma(z). \tag{2}$$

Because the $\Gamma(1) = 1$, then recurrent formula (2) for positive integers n leads to equality [1]

$$\Gamma(n+1) = n\,\Gamma(n) = n\,(n-1)\Gamma(n-2) = \ldots = n(n-1)\cdot\ldots\cdot 2\cdot 1\cdot\Gamma(1),$$

or

$$\Gamma(n+1) = n!,$$

which allows us to consider the gamma function as a generalization of the concept of factorial.

Rewriting the formula (2) in the form,

$$\Gamma(z-1)=\frac{\Gamma(z)}{z-1}, \tag{3}$$

we will obtain an expression that allows us to determine the gamma function from negative arguments for which definition (1) is unacceptable.

Formula (3) shows that $\Gamma(z)$ has at points $z=0,-1,-2,-3...$ ruptures of the second kind.

After repeatedly applying the equality (3) we obtain the formulas for decreasing and increasing, which, respectively, have the form

$$\Gamma(z+n)=z(z+1)...(z+n-1)\Gamma(z), \quad n=1,2,...$$

and

$$\Gamma(z-n)=\frac{\Gamma(z)}{(z-n)(z-n+1)...(z-1)}, \quad n=1,2,...$$

note that

$$\Gamma\left(\frac{1}{2}+n\right)=\frac{(2n)!\sqrt{\pi}}{4^n n!}; \quad \Gamma\left(\frac{1}{2}-n\right)=\frac{(-4)^n n!\sqrt{\pi}}{(2n)!}.$$

The following relations hold: complement formula,

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin z\pi},$$

doubling formula (Legendre's formula)

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right).$$

We also present some values of the Gamma function [11],

$$\left.\begin{array}{ll} \Gamma(n)=(n-1)!, & \Gamma(n+1)=n\Gamma(n), \\ \Gamma(1)=0!=1, & \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}, \\ \Gamma(2)=1!=1, & \Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}, \\ \Gamma(3)=2!=2, & \Gamma\left(\frac{5}{2}\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{3\sqrt{\pi}}{4} \end{array}\right\}. \quad (4)$$

Let's consider solutions to fractional order differential equations using Euler's Gamma functions.

1. Find the solution to the equation $y^{\left(\frac{1}{2}\right)}+\sqrt{x}y=xe^{-x}$.

Multiplying both sides of the equation by e^x we get,

$$y^{\left(\frac{1}{2}\right)}e^x+e^x\sqrt{x}y=x.$$

Let us reduce the right side of the equation to the form $\left(e^x y\right)^{\left(\frac{1}{2}\right)}=x.$ Calculating the order integral $\frac{1}{2}$, using the Riemann-Liouville integral [1], we obtain

$$\left(I_{a+}^{\left(\frac{1}{2}\right)}f\right)(x)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x\frac{t}{\sqrt{x-t}}dt=\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}}(x-p^2)dp=$$

$$=\frac{2}{\sqrt{\pi}}\left(\left(xp-\frac{p^3}{3}\right)\Big|_0^{\sqrt{x}}\right)=\frac{2}{\sqrt{\pi}}\left(\frac{2x\sqrt{x}}{3}\right)=\frac{4x\sqrt{x}}{3\sqrt{\pi}},$$

then the equation will be reduced to the form where $C-$arbitrary constant,

$$ye^x=\frac{4x\sqrt{x}}{3\sqrt{\pi}}+C,$$

solution, which will be written in the form

$$y=\frac{4x\sqrt{x}}{3\sqrt{\pi}}e^{-x}+Ce^{-x}.$$

2. Find a solution to the equation that satisfies the given condition

$$y^{\left(\frac{1}{2}\right)} + y = x^2 + 2x^{\frac{3}{2}}, \qquad y(0) = 0, \qquad \text{let } \Gamma\left(\frac{5}{2}\right) \approx 1.$$

using relation [2] $D^{\alpha+\beta}(\varphi(x)) = D^{\alpha}\left(D^{\beta}(\varphi(x))\right)$, we get

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} y^{\left(\frac{1}{2}\right)} + \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} y = \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} x^2 + 2\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} x^{\frac{3}{2}},$$

$$y' + y^{\left(\frac{1}{2}\right)} = \frac{\Gamma(2+1)}{\Gamma\left(2+1-\frac{1}{2}\right)} x^{2-\frac{1}{2}} + 2\frac{\Gamma\left(\frac{3}{2}+1\right)}{\Gamma\left(\frac{3}{2}+1-\frac{1}{2}\right)} x^{\frac{3}{2}-\frac{1}{2}},$$

equation of the form

$$y' + y^{\left(\frac{1}{2}\right)} = \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)} x^{\frac{3}{2}} + 2\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(2)} x.$$

Considering that $\Gamma\left(\frac{5}{2}\right) \approx 1$ and relations (4),

$$y^{\left(\frac{1}{2}\right)} = -y + x^2 + 2x^{\frac{3}{2}},$$

we obtain an ordinary differential equation of the form,

$$y' - 2x = y - x^2,$$

the solution of which will be written in the form $y = x^2 + Ce^x$, where $C = const$, from the initial condition we get $C = 0$. Then the solution will be written in the form $y = x^2$.

3. Find the solution to the equation,

$$\frac{dy}{dt} + \frac{d^{\frac{1}{2}} y}{dt^{\frac{1}{2}}} - 2y^2 = 0,$$

We look for a solution in the form where $y_0 = c,\ c = const,$

$$y = \sum_{n=0}^{\infty} t^{\frac{n}{2}} y_n,$$

Substituting into the equation we get,

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty} t^{\frac{n}{2}} y_n\right) + \frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}}\left(\sum_{n=0}^{\infty} t^{\frac{n}{2}} y_n\right) - 2\left(\sum_{n=0}^{\infty} t^{\frac{n}{2}} y_n\right)^2 = 0.$$

Using the fractional differentiation formula [1],

$$D^{\alpha}(t^n) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha},$$

we get an equation of the form,

$$\sum_{n=0}^{\infty} \frac{n}{2} t^{\frac{n}{2}-1} y_n + \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)} t^{\frac{n}{2}} y_n - 2\left(\sum_{n=0}^{\infty} t^{\frac{n}{2}} y_n\right)^2 = 0,$$

expanding the series

$$\left(0 + \frac{1}{2} t^{-\frac{1}{2}} y_1 + t^0 y_2 + \frac{3}{2} t^{\frac{1}{2}} y_3 + ...\right) +$$

$$+\left(\frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)} t^{-\frac{1}{2}} y_0 + \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)} t^0 y_1 + \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)} t^{\frac{1}{2}} y_2 + \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(2)} t y_3 + ...\right) -$$

$$-2\left(y_0 + t^{\frac{1}{2}} y_1 + t y_2 + t^{\frac{3}{2}} y_3 + ...\right)^2 = 0.$$

Using (4),

$$\left(0+\frac{1}{2}t^{-\frac{1}{2}}y_1+t^0y_2+\frac{3}{2}t^{\frac{1}{2}}y_3+.......\right)+$$

$$+\left(\frac{1}{\sqrt{\pi}}t^{-\frac{1}{2}}y_0+\frac{\sqrt{\pi}}{2}t^0y_1+\frac{2}{\sqrt{\pi}}t^{\frac{1}{2}}y_2+\frac{3\sqrt{\pi}}{4}ty_3+......\right)-$$

$$-2\left(y_0+t^{\frac{1}{2}}y_1+ty_2+t^{\frac{3}{2}}y_3+.......\right)^2=0,$$

$$t^{-\frac{1}{2}}:\quad \frac{1}{2}y_1+\frac{1}{\sqrt{\pi}}y_0=0,$$

$$t^0\ :\quad y_2+\frac{\sqrt{\pi}}{2}y_1-2y_0{}^2=0,$$

$$t^{\frac{1}{2}}:\quad \frac{3}{2}y_3+\frac{2}{\sqrt{\pi}}y_2-2\left(2y_0y_1\right)=0$$

....

Using $y_0=c$ we get, $y_1=-\frac{2c}{\sqrt{\pi}}$. Substituting it into the second equality we have,

$y_2+\frac{\sqrt{\pi}}{2}\left(-\frac{2c}{\sqrt{\pi}}\right)-2c^2=0,\ \ y_2=2c^2+c.$ Next, substituting into the third equality:

$$y_3=-\frac{8c^2}{\sqrt{\pi}}-\frac{4c}{3\sqrt{\pi}}.$$

The general solution will be written in the form,

$$y=c-\frac{2c}{\sqrt{\pi}}t^{\frac{1}{2}}+\left(2c^2+c\right)t+\left(-\frac{8c^2}{\sqrt{\pi}}-\frac{4c}{3\sqrt{\pi}}\right)t^{\frac{3}{2}}+........$$

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