THE CLASSICAL MITTAG-LEFFLER FUNCTION AND PROPERTIES

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Abstract: In this article, we will present the basic properties of the classical Mittag-Leffler function $E_\alpha(z)$. The material can demonstrate some information starting from the basic definition of the Mittag-Leffler function in terms of a power series, we discover that for parameter α with positive real part the function $E_\alpha(z)$ is an entire function of the complex variable z. Therefore, we discuss in the first part the (analytic) properties of the Mittag-Leffler function as an entire function. As well as, some relations to elementary and special functions before we will show integral representations and differential relations. After that, we calculate integral transforms; lastly, relation to fractional calculus is the last part of the article.

Keywords: analytic, asymptotics, convergence, Cauchy inequality, Taylor coefficients, error function, complementary error function, Hankel path, Wright function, Euler transforms.

1. Introduction and definition

Following Mittag-Leffler's classical definition we consider the one-parametric Mitteg-Leffler function as defined by the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \quad (\alpha \in \mathbb{C}) \tag{1}$$

Applying to the coefficients $c_k = \dfrac{1}{\Gamma(\alpha k+1)}$ of the series (1) the Cauchy-Hadamard formula for the radius of convergence

$$R = \limsup_{k\to\infty} \frac{|c_k|}{|c_{k+1}|} \tag{2}$$

And the asymptotic formula

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1+\frac{(a-b)(a+b+1)}{2z}+O\left(\frac{1}{z^2}\right)\right] \quad (z\to\infty, |\arg z|<\pi) \tag{3}$$

One can see that the series (1) converges in the whole complex plane for all $\mathrm{Re}\,\alpha > 0$. For all $\mathrm{Re}\,\alpha < 0$ it diverges everywhere on $\mathbb{C}\setminus\{0\}$. For $\mathrm{Re}\,\alpha = 0$ the radius of convergence is equal to

$$R = e^{\frac{\pi}{2}|\operatorname{Im}\alpha|}$$

In particular, for $\alpha \in \mathbb{R}_+$ tending to 0 one obtains the following relation:

$$E_0(\pm z) = \sum_{k=0}^{\infty} (\pm 1)^k z^k = \frac{1}{1 \mp z}, \quad |z| < 1 \tag{4}$$

Proposition 1 (Order and type.) *For each* α, $\operatorname{Re}\alpha > 0$, *the Mittag-Leffler function (1) is an entire function of order* $\rho = \frac{1}{\operatorname{Re}\alpha}$ *and type* $\sigma = 1$.

One can also observe that from the above Proposition 1.1 it follows that the function

$$E_\alpha(\sigma^\alpha z) = \sum_{k=0}^{\infty} \frac{(\sigma^\alpha z)^k}{\Gamma(\alpha k + 1)}, \quad \sigma > 0$$

has order $\rho = \frac{1}{\operatorname{Re}\alpha}$ and type σ.

2.Relations to Elementary and Special Functions

The Mittag-Leffler functions plays an important role among special functions. First of all, it is not difficult to obtain a number of its relations to elementary and special functions. The simplest relation is formula (4) representing $E_0(z)$ as the sum of a geometric series.

Proposition 2 (Special cases.) For all $z \in \mathbb{C}$ the Mittag-Leffler function satisfies the following relations

$$E_1(\pm z) = \sum_{k=0}^{\infty} \frac{(\pm 1)^k z^k}{\Gamma(k+1)} = e^{\pm z} \tag{5}$$

$$E_2(-z^2) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(2k+1)} = \cos z \tag{6}$$

$$E_2(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \cosh z \tag{7}$$

Here we can use Taylor series of a function

$$f(z) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots.$$

$$E_{\frac{1}{2}}(\pm z^{\frac{1}{2}}) = \sum_{k=0}^{\infty} \frac{(\pm 1)^k z^{\frac{k}{2}}}{\Gamma\left(\frac{1}{2}k + 1\right)} = e^z[1 + \operatorname{erf}(\pm z^{\frac{1}{2}})] = e^z \operatorname{erfc}(\mp z^{\frac{1}{2}}) \tag{8}$$

where erf (erfc) denotes the error function (complementary error function)

$$\operatorname{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}du,\ \operatorname{erfc}(z)=1-erf(z),\ z\in\mathbb{C}$$

and $z^{\frac{1}{2}}$ means the principle branch of the corresponding multi-valued function defined in the whole complex plane cut along the negative real semi-axis.

A more general formula for the function with half-integer parameter is valid

$$E_{\frac{p}{2}}(z)={}_0F_{p-1}\left(;\frac{1}{p},\frac{2}{p},\ldots,\frac{p-1}{p};\frac{z^2}{p^p}\right)+ \\ +\frac{2^{\frac{p+1}{2}}z}{p!\sqrt{\pi}}{}_1F_{2p-1}\left(1;\frac{p+2}{2p},\frac{p+3}{2p},\ldots,\frac{3p}{2p};\frac{z^2}{p^p}\right) \tag{9}$$

where ${}_pF_q$ is the (p,q) – hypergeometric function

$${}_pF_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_p;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\ldots(a_p)_k}{(b_1)_k(b_2)_k\ldots(b_q)_k}\frac{z^k}{k!} \tag{10}$$

where $(a)_k=a(a+1)(a+2)\ldots(a+k-1)$ – Pochhammer symbol.

Let us prove formula (8). We first rewrite the series representation (1) assuming that $z^{\frac{1}{2}}$ is the principle branch of the corresponding multi-valued function and substituting z in place of $z^{\frac{1}{2}}$:

$$E_{\frac{1}{2}}(z)=\sum_{m=0}^{\infty}\frac{z^{2m}}{\Gamma(m+1)}+\sum_{m=0}^{\infty}\frac{z^{2m+1}}{\Gamma\left(m+\frac{3}{2}\right)}=u(z)+v(z) \tag{11}$$

The sum $u(z)$ is equal to e^{z^2}. To obtain the formula for the remaining function v one can use the series representation of the error function:

$$erf(z)=\frac{2}{\sqrt{\pi}}e^{-z^2}\sum_{m=0}^{\infty}\frac{2^m}{(2m+1)!!}z^{2m+1},\ z\in\mathbb{C} \tag{12}$$

An alternative proof can be obtained by a term-wise differentiation of the second series in (11). It follows that $v(z)$ satisfies the Cauchy problem for the first-order differential equation in $\mathbb{C}$.

$$v'(z)=2\left[\frac{1}{\sqrt{\pi}}+zv(z)\right],\ v(0)=0$$

Representation (8) follows from the solution of this problem

$$v(z)=e^{z^2}\frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}du=e^{z^2}\operatorname{erf}(z)$$

3. Recurrence and Differential Relations

Proposition 3 (Recurrence relations.) The following recurrence formulas relating the Mittag-Leffler function for different values of parameters hold:

$$E_{p/q}(z)=\frac{1}{q}\sum_{l=0}^{q-1}E_{1/p}\left(z^{1/q}e^{\frac{2\pi li}{q}}\right),\ q\in\mathbb{N} \tag{13}$$

$$E_{\frac{1}{q}}\left(z^{\frac{1}{q}}\right)=e^{z}\left[1+\sum_{m=0}^{q-1}\frac{\gamma\left(1+\frac{m}{q},z\right)}{\Gamma\left(1-\frac{m}{q}\right)}\right],\ q=2,3,\ldots \tag{14}$$

where $\gamma(a,z)=\int_0^z e^{-u}u^{a-1}du$ denotes the incomplete gamma function, and $z^{1/q}$ means the principal branch of the corresponding multi-valued function.

◁ To prove relation (13) we use the well-known identity (discrete orthogonality relation)

$$\sum_{l=0}^{p-1}e^{\frac{2\pi lki}{p}}=\begin{cases}p, & if\ k\equiv 0\,(\mathrm{mod}\ p),\\ 0, & if\ k\neq 0\,(\mathrm{mod}\ p).\end{cases} \tag{15}$$

This together with definition (1) of the Mittag-Leffler function gives

$$\sum_{l=0}^{p-1}E_{\alpha}(ze^{\frac{2\pi li}{p}})=pE_{\alpha\cdot p}(z^{p}),\ p\geq 1. \tag{16}$$

Substituting $\frac{\alpha}{p}$ for α and $z^{\frac{1}{2}}$ for z we arrive at the desired relation (1) after setting $\alpha=p/q$.

We mention the following “symmetric” variant of (16):

$$E_{\alpha}(z)=\frac{1}{2m+1}\sum_{l=-m}^{m}E_{\alpha/(2m+1)}\left(z^{\frac{1}{2m+1}}e^{\frac{2\pi li}{2m+1}}\right),\ m\geq 0 \tag{17}$$

Relation (14) follows by differentiation, valid for all $p,q\in\mathbb{N}$.

Proposition 4 (Differential relations.)

$$\left(\frac{d}{dz}\right)^{p}E_{p}\left(z^{p}\right)=E_{p}\left(z^{p}\right) \tag{18}$$

$$\frac{d^{p}}{dz^{p}}E_{p/q}\left(z^{p/q}\right)=E_{p/q}\left(z^{p/q}\right)+\sum_{k=1}^{q-1}\frac{z^{-kp/q}}{\Gamma(1-kp/q)},\ q=2,3,\ldots. \tag{19}$$

Let $p=1$ in (19). Multiplying both sides of the corresponding relation by e^{-z} we get

$$\frac{d}{dz}[e^{-z}E_{1/q}(z^{1/q})]=e^{-z}\sum_{k=1}^{q-1}\frac{z^{-k/q}}{\Gamma(1-k/q)}E_{p/q}$$

By integrating and using the definition of the incomplete gamma function we arrive at the relation (19). The relation (19) shows that the Mittag-Leffler functions of rational order can be expressed in terms of exponentials and incomplete gamma function. In particular, for $q=2$ we obtain the relation

$$E_{1/2}\left(z^{1/2}\right)=e^{z}\left[1+\frac{1}{\sqrt{\pi}}\gamma\left(1/2,z\right)\right] \tag{20}$$

This is equivalent to relation (16) by the formula $\operatorname{erf}\left(z\right)=\dfrac{\gamma\left(1/2,z^{2}\right)}{\sqrt{\pi}}$.

4. Integral Representations

Many important properties of the Mittag-Leffler function follow from its integral representations. Let us denote by $\gamma\left(\varepsilon;a\right)\left(\varepsilon>0,0<a\leq\pi\right)$ a contour oriented by non-decreasing $\arg\zeta$ consisting of the following parts: the ray $\arg\zeta=-a,\left|\zeta\right|\geq\varepsilon$, the arc $-a\leq\arg\zeta\leq a,\left|\zeta\right|=\varepsilon$, and the ray $\arg\zeta=a,\left|\zeta\right|\geq\varepsilon$. If $0<a<\pi$, then the contour $\gamma\left(\varepsilon;a\right)$ divides the complex ζ -plane into two unbounded parts, namely $G^{(-)}\left(\varepsilon;a\right)$ to the left of $\gamma\left(\varepsilon;a\right)$ by orientation, and $G^{(+)}\left(\varepsilon;a\right)$ to the right of it. If $a=\pi$, then the contour consists of the circle $\left|\zeta\right|=\varepsilon$ and the twice passable ray $-\infty<\zeta\leq-\varepsilon$. In both cases the contour $\gamma\left(\varepsilon;a\right)$ is called the *Hankel path* (as it is used in the representation of the reciprocal of the Gamma function).

Lemma 1 Let $0<\alpha<2$ and

$$\frac{\pi\alpha}{2}<\beta\leq\min\left\{\pi,\pi\alpha\right\} \tag{21}$$

Then the Mittag-Leffler function can be represented in the form

$$E_{\alpha}\left(z\right)=\frac{1}{2\pi\alpha i}\int_{\gamma(\varepsilon;\beta)}\frac{e^{\zeta^{1/\alpha}}}{\zeta-z}d\zeta,\ z\in G^{(-)}\left(\varepsilon;\beta\right); \tag{22}$$

$$E_{\alpha}\left(z\right)=\frac{1}{\alpha}e^{z^{1/\alpha}}+\frac{1}{2\pi\alpha i}\int_{\gamma(\varepsilon;\beta)}\frac{e^{\zeta^{1/\alpha}}}{\zeta-z}d\zeta,\ z\in G^{(+)}\left(\varepsilon;\beta\right) \tag{23}$$

Let $\alpha=2$. Then the Mittag-Leffler function E_2 can be represented in the form

$$E_{2}\left(z\right)=\frac{1}{4\pi i}\int_{\gamma(\varepsilon;\pi)}\frac{e^{\zeta^{1/2}}}{\zeta-z}d\zeta,\ z\in G^{(-)}\left(\varepsilon;\pi\right); \tag{24}$$

$$E_{2}\left(z\right)=\frac{1}{2}e^{z^{1/2}}+\frac{1}{4\pi i}\int_{\gamma(\varepsilon;\pi)}\frac{e^{\zeta^{1/2}}}{\zeta-z}d\zeta,\ z\in G^{(+)}\left(\varepsilon;\pi\right) \tag{25}$$

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In (22)-(23) the function $z^{\frac{1}{\alpha}}$ (or $\zeta^{\frac{1}{\alpha}}$) means the principal branch of the corresponding multi-valued function determined in the complex plane $\mathbb{C}$ cut along the negative semi-axis which is positive for positive z (respectively, ζ).

◁ We use in the proof the Hankel integral representation (or Hankel's formula) for the reciprocal of the Euler Gamma function

$$\frac{1}{\Gamma(s)}=\frac{1}{2\pi i}\int_{\gamma(\varepsilon;a)} e^u u^{-s}du,\ \varepsilon>0,\frac{\pi}{2}<a<\pi,s\in\mathbb{C} \tag{26}$$

Formula (26) is also valid for $a=\frac{\pi}{2}$, $\operatorname{Re}s>0$, i.e.

$$\frac{1}{\Gamma(s)}=\frac{1}{2\pi i}\int_{\gamma\left(\varepsilon;\frac{\pi}{2}\right)} e^u u^{-s}du,\ \varepsilon>0,\operatorname{Re}s>0. \tag{27}$$

We know rewrite formulas (26) and (27) in a slightly modified form. Let us begin with the integral representation (26). After the change of variables $u=\zeta^{1/\alpha}$ (in the case $1\le\alpha<2$ we only consider the contours $\gamma(\varepsilon;\theta)$ with $\theta\in(\pi/2,\pi/\alpha)$) we arrive at

$$\frac{1}{\Gamma(s)}=\frac{1}{2\pi i\alpha}\int_{\gamma(\varepsilon;\beta)} e^{\zeta^{1/\alpha}}\zeta^{\frac{-s+1}{\alpha}-1}d\zeta,\ \pi\alpha/2<\beta\le\min\{\pi,\pi\alpha\} \tag{28}$$

Similarly, using the change of variables $u=\zeta^{1/2}$ in (27) we have

$$\frac{1}{\Gamma(s)}=\frac{1}{4\pi i}\int_{\gamma(\varepsilon;\pi)} e^{\zeta^{1/2}}\zeta^{\frac{-s+1}{2}}d\zeta,\operatorname{Re}s>0 \tag{29}$$

Let us begin with the case $\alpha<2$. First let $|z|<\varepsilon$. In this case

$$\sup_{\zeta\in\gamma(\varepsilon;\beta)}\left|z\zeta^{-1}\right|<1$$

It now follows from the integral representation (29) and the definition (1) of the function $E_\alpha(z)$ that for $0<\alpha<2,|z|<\varepsilon$,

$$E_a(z)=\sum_{k=0}^{\infty}\frac{1}{2\pi i\alpha}\left\{\int_{\gamma(\varepsilon;\beta)} e^{\zeta^{1/\alpha}}\zeta^{\frac{-\alpha k-1+1}{\alpha}-1}d\zeta\right\}z^k$$

$$=\frac{1}{2\pi i\alpha}\int_{\gamma(\varepsilon;\beta)} e^{\zeta^{1/\alpha}}\frac{1}{\zeta}\left\{\sum_{k=0}^{\infty}\left(z\zeta^{-1}\right)^k\right\}d\zeta$$

$$=\frac{1}{2\pi i\alpha}\int_{\gamma(\varepsilon;\beta)}\frac{e^{\zeta^{1/\alpha}}}{\zeta-z}d\zeta$$

The last integral converges absolutely under condition (21) and represent an analytic function of z in each of the two domains: $G^{(-)}(\varepsilon;\beta)$ and $G^{(+)}(\varepsilon;\beta)$. On the other hand, the disk $|z|<\varepsilon$ is contained in the domain $G^{(-)}(\varepsilon;\beta)$ for any β. It follows from the Analytic Continuation Principle that the integral representation (22) holds for the whole domain $G^{(-)}(\varepsilon;\beta)$.

Let now $z \in G^{(+)}(\varepsilon;\beta)$. Then for any $\varepsilon_1 > |z|$ we have $z \in G^{(-)}(\varepsilon;\beta)$, and using formula (22) we arrive at

$$E_\alpha(z) = \frac{1}{2\pi i\alpha}\int_{\gamma(\varepsilon_1;\beta)} \frac{e^{\zeta^{1/\alpha}}}{\zeta - z} d\zeta \qquad (30)$$

On the other hand, for $\varepsilon < |z| < \varepsilon_1$, $|\arg z| < \beta$, it follows from the Cauchy integral theorem that

$$\frac{1}{2\pi i\alpha}\int_{\gamma(\varepsilon_1,\beta)-\gamma(\varepsilon;\beta)} \frac{e^{\zeta^{1/\alpha}}}{\zeta - z} d\zeta = \frac{1}{\alpha} e^{z^{1/\alpha}} \qquad (31)$$

The representation (23) of the function $E_\alpha(z)$ in the domain $G^{(+)}(\varepsilon;\beta)$ now follows from (30) and (31).

To prove the integral representations (24) and (25) for $\alpha = 2$ we argue analogously to the case $0 < \alpha < 2$ using the representation (29). Recall that there is no need to revise formula (25) for $\alpha = 2$ since we have exact representations (6) and (7) in this case.

It should be noted that integral representations (22)-(23) can be used for the representation of the function $E_\alpha(z)$, $0 < \alpha < 2$, at any point z of the complex plane. To obtain such a representation it is sufficient to consider contours $\gamma(\varepsilon;\beta)$ and $\gamma(\varepsilon;\pi)$ with parameter $\varepsilon < |z|$.

The above given representations (22)-(25) can be rewritten in a unique form, namely in the form of the classical *Mittag-Leffler integral representation*

$$E_\alpha(z) = \frac{1}{2\pi i}\int_{Ha_-} \frac{\zeta^{\alpha-1} e^{\zeta}}{\zeta^{\alpha} - z} d\zeta \qquad (32)$$

Where the path of integration Ha_ is a loop which starts and ends at $-\infty$ approaching along the negative semi-axis and encircles the disk $|\zeta| \le |z|^{1/\alpha}$ in the positive sense: $-\pi \le \arg\zeta \le \pi$ on Ha_.

5. Conclusion

Mittag-Leffler functions with one parameter functions as a fundamental base in learning and developing next generations of this function, Mittag-Leffler function with two variables and Prabhakar or Kilbas-Saigo type. The principal object of this work is to present a natural further step toward the mathematical properties concerning the Mittag-Leffler function with two variables which is more advanced version of this Mittag-Leffler function. Also, the standard Poison process is generalized bt replacing the exponential function (as a waiting time density) by the Mittag-Leffler function (taking into account its power law behavior at infinity). The corresponding renewal process. By taking the diffusion limit of the Mittag-Leffler renewal process one can arrive at the space-time fractional diffusion equation.

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